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SOLVABILITY OF MULTI-POINT BOUNDARY VALUE PROBLEMS FOR FRACTIONAL DIFFERENTIAL EQUATIONS AT RESONANCE

Yuji Liu* and Xingyuan Liu**

Abstract. Sufficient conditions for the existence of at least one solution of a class of multi-point boundary value problems of the fractional differential equations at resonance are established. The main theorem generalizes and improves those ones in [Liu, B., Solvability of multi-point boundary value problems at resonance(II), Appl. Math. Comput., 136(2003)353-377], see Remark 2.3. An example is presented to illustrate the main results.

1. Introduction

Fractional differential equations have many applications in modeling of physical and chemical processes [1-8]. In its turn, mathematical aspects of fractional differential equations and methods of their solutions were discussed by many authors, see the text books [3,5], the papers [1,2,4,6-8] and the references therein.

However, there are not many papers consider the boundary value problems at resonance for nonlinear ordinary differential equations of fractional order.

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Correspondence should be addressed to Yuji Liu, liuyuji888@sohu.com.

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In this paper, we discuss the multi-point boundary value problem (MBVP or BVP for short) of the nonlinear fractional differential equation (FDE for short) with the nonlinearity depending on $D_{0+}^{\alpha-1}u$

$$(1.1)\qquad \begin{cases} D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\alpha-1}u(t)\right) + e(t), t \in (0,1), \\ \lim_{t\to 0} D_{0+}^{\alpha-1}u(t) = 0, \\ \lim_{t\to 1} D_{0+}^{\alpha-1}u(t) = \sum_{i=1}^{m-2} \beta_i D_{0+}^{\alpha-1}u(\xi_i), \end{cases}$$

where D_{0+}^{α} is the Riemann-Liouville fractional derivative of order $\alpha \in (1,2)$, and $f : (0,1) \times R \times R \to R$ is continuous, $e \in L^1(0,1)$, f and e may be singular at $t = 0$ and $t = 1$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $\beta_i \in R(i = 1, 2, \cdots, m-2)$ are constants. We obtain the results on the existence of solutions of BVP(1.1) by using the coincidence degree theory in Banach spaces.

One sees that the corresponding homogeneous boundary value problem of BVP(1.1) is as follows:

$$\begin{cases} D_{0+}^{\alpha}u(t) = 0, t \in (0,1), \\ \lim_{t\to 0} D_{0+}^{\alpha-1}u(t) = 0, \\ \lim_{t\to 1} D_{0+}^{\alpha-1}u(t) = \sum_{i=1}^{m-2} \beta_i D_{0+}^{\alpha-1}u(\xi_i). \end{cases}$$

It has nontrivial solutions $u(t) = ct^{\alpha-2}$, $c \in R$. Hence BVP(1.1) is called a resonant boundary value problem.

It is easy to see when $\alpha = 2$ that BVP(1.1) becomes the multi-point BVPs

$$(1.2)\qquad \begin{cases} u''(t) = f(t, u(t), u'(t)) + e(t), \quad t \in (0,1), \\ u'(0) = 0, \\ u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i), \end{cases}$$

where $f : [0,1] \times R \times R \to R$ is continuous, $e \in L^1[0,1]$, $0 < \xi_1 < \xi_2 < \cdots < \xi_{m-2} < 1$ and $\beta_i \in R(i = 1, 2, \cdots, m-2)$ are constants. BVP(1.2) has been studied by many authors, see [9,10]. The purpose of this paper is to generalize parts of the results obtained in [9,10].

2. Main results

To obtain the main results, we need some notations and an abstract existence theorem by Gaines and Mawhin [11].

Let X and Y be Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm operator of index zero, $P : X \to X$, $Q : Y \to Y$ be projectors such that

$$\text{Im } P = \text{Ker } L, \ \text{Ker } Q = \text{Im } L, \ \ X = \text{Ker } L \oplus \text{Ker } P, \ \ Y = \text{Im } L \oplus \text{Im } Q.$$

It follows that

$$L|_{D(L)\cap \mathrm{Ker}\, P}: \ D(L)\cap \mathrm{Ker}\, P \to \mathrm{Im}\, L$$

is invertible, we denote the inverse of that map by K_p.

If Ω is an open bounded subset of X, $D(L)\cap\overline{\Omega} \neq \emptyset$, the map $N: \ X \to Y$ will be called $L-$compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N: \overline{\Omega} \to X$ is compact.

LEMMA 2.1. [11] *Let L be a Fredholm operator of index zero and let N be $L-$compact on Ω. Assume that the following conditions are satisfied:*

(i) *$Lx \neq \lambda Nx$ for every $(x,\lambda) \in [D(L) \setminus KerL) \cap \partial\Omega] \times (0,1)$;*
(ii) *$Nx \notin ImL$ for every $x \in KerL \cap \partial\Omega$;*
(iii) *$deg(\wedge^{-1}QN\,\big|_{KerL},\ \Omega\cap KerL, 0) \neq 0$, where $\wedge^{-1}: \ Y/ImL \to KerL$ is an isomorphism.*

Then the equation $Lx = Nx$ has at least one solution in $D(L)\cap\overline{\Omega}$.

We use the Banach spaces $C[0,1]$ with the norm

$$||u||_\infty = \max_{t\in[0,1]} |u(t)|,$$

and $L^1[0,1]$ with the norm

$$||u||_1 = \int_0^1 |u(s)|ds.$$

Let

$$X = \left\{ u \in C^0(0,1): \begin{array}{c} D_{0^+}^{\alpha-1}x \in C^0(0,1) \\ \text{there exist the limits} \\ \lim_{t\to 0} t^{2-\alpha}x(t) \\ \lim_{t\to 1} t^{2-\alpha}x(t) \\ \lim_{t\to 0} D_{0^+}^{\alpha-1}x(t) \\ \lim_{t\to 1} D_{0^+}^{\alpha-1}x(t) \end{array} \right\}.$$

For $x \in X$, define the norm

$$||x|| = \max\left\{ \sup_{t\in(0,1)} t^{2-\alpha}|x(t)|, \ \sup_{t\in(0,1)} |D_{0^+}^{\alpha-1}x(t)| \right\}.$$

By means of the linear functional analysis theory, we can prove that X is a Banach space. Choose $Y = L^1[0,1]$.

Define L to be the linear operator from $D(L)\bigcap X$ to Y with

$$D(L)=\left\{u\in C^{\alpha-1}[0,1]:\begin{array}{c}D_{0+}^{\alpha}u\in L^1[0,1],\\ \lim_{t\to 0}D_{0+}^{\alpha-1}u(t)=0,\\ \lim_{t\to 1}D_{0+}^{\alpha-1}u(t)=\sum_{i=1}^{m-2}\beta_iD_{0+}^{\alpha-1}u(\xi_i)\end{array}\right\}$$

and

$$(Lu)(t)=D_{0+}^{\alpha}u(t),\ u\in D(L).$$

Define $N:X\to Y$ by

$$(Nu)(t)=f\left(t,u(t),D_{0+}^{\alpha-1}u(t)\right)+e(t),\ u\in X.$$

Then BVP(1.1) can be written as

$$Lu=Nu,\ \ u\in D(L).$$

Lemma 2.2. *It holds that*

(i) *$KerL=\{ct^{\alpha-2},\ c\in R\}$;*

(ii) *$ImL=\left\{\ v\in Y,\ \int_0^1v(s)ds=\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds\ \right\}$;*

(iii) *L is a Fredholm operator of index zero;*

(iv) *There exist projectors $P:X\to X$ and $Q:Y\to Y$ such that $KerL=ImP$ and $KerQ=ImL$. Furthermore, let $\Omega\subset X$ be an open bounded subset with $\overline{\Omega}\cap D(L)\neq\emptyset$, then N is $L-$compact on $\overline{\Omega}$.*

Proof. One sees that $D_{0+}^{\alpha}u(t)=0$ has solutions

$$u(t)=c_1t^{\alpha-1}+c_2t^{\alpha-2},t\in(0,1)$$

for some $c_i\in R,\ \ i=1,2$. We get

$$D_{0+}^{\alpha-1}u(t)=\Gamma(\alpha)c_1.$$

It follows from

$$\lim_{t\to 0}D_{0+}^{\alpha-1}u(t)=0,\ \lim_{t\to 1}D_{0+}^{\alpha-1}u(t)=\sum_{i=1}^{m-2}\beta_iD_{0+}^{\alpha-1}u(\xi_i)$$

that $c_1=0$ and $c_2\in R$. Then **(i)** follows immediately.

We see that $v\in$ ImL if and only if there exists a function $u\in D(L)$ such that

$$\begin{cases}D_{0+}^{\alpha}u(t)=v(t),\quad t\in(0,1),1<\alpha\leq 2,\\ \lim_{t\to 0}D_{0+}^{\alpha-1}u(t)=0,\\ \lim_{t\to 1}D_{0+}^{\alpha-1}u(t)=\sum_{i=1}^{m-2}\beta_iD_{0+}^{\alpha-1}u(\xi_i).\end{cases}$$

Then

$$u(t)=I_{0+}^{\alpha}v(t)+c_1t^{\alpha-1}+c_2t^{\alpha-2},t\in(0,1).\tag{2.1}$$

It follows that

$$D_{0^+}^{\alpha-1}u(t) = \int_0^t v(s)ds + c_1\Gamma(\alpha). \tag{2.2}$$

From the boundary conditions, we get $c_1 = 0$ and

$$\int_0^1 v(s)ds = \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} v(s)ds. \tag{2.3}$$

On the other hand, suppose $v \in Y$ and satisfies (2.3). Choose

$$u(t) = I_{0^+}^{\alpha} v(t).$$

One sees by computation that

$$D_{0^+}^{\alpha} u(t) = v(t), t \in (0,1)$$

and

$$D_{0^+}^{\alpha-1}u(t) = \int_0^t v(s)ds.$$

One has

$$\lim_{t\to 0} D_{0^+}^{\alpha-1}u(t) = 0,\ \lim_{t\to 1} D_{0^+}^{\alpha-1}u(t) = \sum_{i=1}^{m-2} \beta_i D_{0^+}^{\alpha-1}u(\xi_i).$$

Furthermore, we know that $u \in C(0,1)$ and $D_{0^+}^{\alpha-1}u \in C(0,1)$ and there exist the limits

$$\lim_{t\to 0} t^{2-\alpha}x(t),\ \lim_{t\to 1} t^{2-\alpha}x(t),\ \lim_{t\to 0} D_{0^+}^{\alpha-1}x(t),\ \lim_{t\to 1} D_{0^+}^{\alpha-1}x(t).$$

Hence $u \in D(L)$ and $Lu = v$. So $v \in \mathrm{Im}L$. Then **(ii)** follows.

To prove (**iii**) and (**iv**), we first claim that there exists $k \in \{0, 1, 2, \cdots, m-2\}$ such that $\sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha+k} \neq 1$. In fact, suppose that

$$\sum_{i=1}^{m-2} \beta_i \xi_i^{\alpha+k} = 1 \text{ for all } k \in \{0, 1, 2, \cdots, m-2\},$$

we have

$$\begin{pmatrix} \xi_1^{\alpha} & \xi_2^{\alpha} & \cdots & \xi_{m-2}^{\alpha} \\ \xi_1^{\alpha+1} & \xi_2^{\alpha+1} & \cdots & \xi_{m-2}^{\alpha+1} \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdots & \cdot \\ \xi_1^{\alpha+m-2} & \xi_2^{\alpha+m-2} & \cdots & \xi_{m-2}^{\alpha+m-2} \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ \cdot \\ \cdot \\ \cdot \\ \beta_{m-2} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}.$$

It is equal to

$$\begin{pmatrix} \xi_1^{\alpha} & \xi_2^{\alpha} & \cdots & \xi_{m-2}^{\alpha} & 1 \\ \xi_1^{\alpha+1} & \xi_2^{\alpha+1} & \cdots & \xi_{m-2}^{\alpha+1} & 1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & .. & \\ \xi_1^{\alpha+m-2} & \xi_2^{\alpha+m-2} & \cdots & \xi_{m-2}^{\alpha+m-2} & 1 \end{pmatrix} \begin{pmatrix} \beta_1 \\ \beta_2 \\ . \\ . \\ . \\ \beta_{m-2} \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \\ 0 \end{pmatrix}.$$

However, it is well known that the Vandermont Determinant

$$\begin{vmatrix} \xi_1^{\alpha} & \xi_2^{\alpha} & \cdots & \xi_{m-2}^{\alpha} & 1 \\ \xi_1^{\alpha+1} & \xi_2^{\alpha+1} & \cdots & \xi_{m-2}^{\alpha+1} & 1 \\ . & . & \cdots & . & . \\ . & . & \cdots & . & . \\ . & . & \cdots & .. & \\ \xi_1^{\alpha+m-2} & \xi_2^{\alpha+m-2} & \cdots & \xi_{m-2}^{\alpha+m-2} & 1 \end{vmatrix}$$

is not equal to zero, so there is a contradiction.

Let k satisfy $\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k} \neq 1$. Define the projectors $Q: \ Y \to Y$ and $P: \ X \to X$ by

$$(Qv)(t) = (\alpha+k)\frac{\int_0^1 v(s)ds - \sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i} v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}t^{\alpha+k-1} \ \text{ for } \ v \in Y,$$

and

$$(Pu)(t) = \frac{\lim_{t\to 0} t^{2-\alpha}u(t)}{\Gamma(\alpha-1)}t^{\alpha-2} \ \text{ for } \ u \in X,$$

respectively. It is easy to prove that $\text{Im } P = \text{Ker } L, \ \text{Ker } Q = \text{Im } L.$

Furthermore, for $u \in X$, one sees

$$\frac{\lim_{t\to 0} t^{2-\alpha}u(t)}{\Gamma(\alpha-1)}t^{\alpha-2} \in \text{Ker}L,$$

and the definition of P implies

$$\begin{aligned} & P\left(u(t) - \frac{\lim_{t\to 0} t^{2-\alpha}u(t)}{\Gamma(\alpha-1)}t^{\alpha-2}\right) \\ = \ & Pu(t) - P\left(\frac{\lim_{t\to 0} t^{2-\alpha}u(t)}{\Gamma(\alpha-1)}t^{\alpha-2}\right) \\ = \ & \frac{\lim_{t\to 0} t^{2-\alpha}u(t)}{\Gamma(\alpha-1)}t^{\alpha-2} - \frac{\lim_{t\to 0} t^{2-\alpha}u(t)}{\Gamma(\alpha-1)}P\left(t^{\alpha-2}\right) = 0. \end{aligned}$$

We get

$$u(t)-\frac{\lim_{t\to 0}t^{2-\alpha}u(t)}{\Gamma(\alpha-1)}t^{\alpha-2}\in \mathrm{Ker}P.$$

One can see that $\mathrm{Ker}L\bigcap \mathrm{Ker}P=\{0\}$. Then $X=\mathrm{Ker}\,L\oplus\mathrm{Ker}\,P$.

For $v\in Y$, since

$$\begin{aligned}
&\int_0^1\left(v-(\alpha+k)\frac{\int_0^1 v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}t^{\alpha+k-1}\right)dt\\
=&\int_0^1 v(s)ds-(\alpha+k)\frac{\int_0^1 v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}\int_0^1 t^{\alpha+k-1}dt\\
=&\int_0^1 v(s)ds-\frac{\int_0^1 v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}},
\end{aligned}$$

implies

$$\begin{aligned}
&\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\left(v-(\alpha+k)\frac{\int_0^1 v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}t^{\alpha+k-1}\right)dt\\
=&\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(t)dt\\
&-\frac{\int_0^1 v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}\sum_{i=1}^{m-2}\beta_i(\alpha+k)\int_0^{\xi_i}t^{\alpha+k-1}dt\\
=&\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(t)dt-\frac{\int_0^1 v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}\\
=&\int_0^1 v(s)ds-\frac{\int_0^1 v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}\\
=&\int_0^1\left(v-(\alpha+k)\frac{\int_0^1 v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}t^{\alpha+k-1}\right)dt,
\end{aligned}$$

we get

$$v-(\alpha+k)\frac{\int_0^1 v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}t^{\alpha+k-1}\in \mathrm{Im}L.$$

Together with

$$(\alpha+k)\frac{\int_0^1 v(s)ds-\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}v(s)ds}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}t^{\alpha+k-1}\in \mathrm{Im}Q,$$

and $\mathrm{Im}L \bigcap \mathrm{Im}Q = \{0\}$, so $Y = \mathrm{Im}\ L \oplus \mathrm{Im}\ Q$. It follows that $Y/\mathrm{Im}L = \mathrm{Im}Q$. So $\dim \mathrm{Ker}L = \dim Y/\mathrm{Im}L = 1$. Hence L is a Fredholm operator of index zero.

For $v \in \mathrm{Im}L$, let

$$(K_P v)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds = I_{0+}^{\alpha} v(t) \quad \text{for} \ \ v \in \mathrm{Im}L.$$

One sees $K_P v \in D(L)$ and

$$\begin{aligned} & P\left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds\right) \\ & = P(I_{0+}^{\alpha} v(t)) = \frac{\lim_{t\to 0} t^{2-\alpha} I_{0+}^{\alpha} v(t)}{\Gamma(\alpha-1)} t^{\alpha-2} = 0. \end{aligned}$$

It follows that $(K_p v) \in \mathrm{Ker}P$. Then $K_P :\ \mathrm{Im}\ L \to D(L) \cap \mathrm{Ker}P$ is well defined.

Furthermore, for $v \in \mathrm{Im}L$, we have

$$(LK_P)(v) = L\left(\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds\right) = D_{0+}^{\alpha} \left(I_{0+}^{\alpha} v(t)\right) = v(t).$$

On the other hand, for $u \in \mathrm{Ker}\ P \cap D(L)$, we have

$$(Pu)(t) = \frac{\lim_{t\to 0} t^{2-\alpha} u(t)}{\Gamma(\alpha-1)} t^{\alpha-2} = 0, t \in (0,1).$$

Then $\lim_{t\to 0} t^{2-\alpha} u(t) = 0$. Suppose $D_{0+}^{\alpha} u = v$. Then

$$u(t) \ = \ I_{0+}^{\alpha} v(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}.$$

Since $D_{0+}^{\alpha-1} u(t) = 0$ and $\lim_{t\to 0} t^{2-\alpha} u(t) = 0$, then $c_1 = c_2 = 0$. It follows from the definition of K_P that

$$\begin{aligned} (K_P L)u(t) &= K_P D_{0+}^{\alpha} u(t) = K_P v(t) \\ &= \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} v(s)ds \\ &= u(t). \end{aligned}$$

Then K_P is the inverse of $L : D(L) \bigcap \mathrm{Ker}P \to \mathrm{Im}L$. The isomorphism $\wedge :\ \mathrm{Ker}L \to Y/\mathrm{Im}L$ is given by

$$\wedge(at^{\alpha-2}) = at^{\alpha+k-1}.$$

Furthermore, one has

$$
\begin{aligned}
& QNu(t) \\
= \; & Q\left(f(t,u(t),D_{0^+}^{\alpha-1}u(t))+e(t)\right) \\
= \; & \frac{(\alpha+k)t^{\alpha+k-1}}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}\int_0^1\left(f(t,u(t),D_{0^+}^{\alpha-1}u(t))+e(t)\right)dt \\
& -\frac{(\alpha+k)t^{\alpha+k-1}}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\left(f(t,u(t),D_{0^+}^{\alpha-1}u(t))+e(t)\right)dt, \\
& K_p(I-Q)Nx(t) \\
= \; & K_p(I-Q)\left(f(t,u(t),D_{0^+}^{\alpha-1}u(t))+e(t)\right) \\
= \; & K_P\left(f(t,u(t),D_{0^+}^{\alpha-1}u(t))+e(t)\right) \\
& -K_PQ\left(f(t,u(t),D_{0^+}^{\alpha-1}u(t))+e(t)\right) \\
= \; & \frac{1}{\Gamma(\alpha)}\int_0^t(t-s)^{\alpha-1}\left(f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s)\right)ds \\
& -\frac{1}{\Gamma(\alpha)}\frac{\alpha+k}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}}\int_0^1\left(f(t,u(t),D_{0^+}^{\alpha-1}u(t))+e(t)\right)dt\times \\
& \int_0^t(t-s)^{\alpha-1}s^{\alpha+k-1}ds.
\end{aligned}
$$

Since f is continuous, Let Ω be a bounded subset in Y. It is easy to show that $QN(\Omega)$ and $K_P(I-Q)N(\Omega)$ are bounded in Y, $K_P(I-Q)N(\Omega)$ and $D_{0^+}^{\alpha-1}K_P(I-Q)N(\Omega)$ are equicontinuous. Then $K_P(I-Q)N$ is completely continuous. So N is $L-$compact on Ω. The proofs are completed. $\square$

Theorem 2.3. *Suppose*

(A) *there exist nonnegative continuous functions $a,b,c,r\in L^1(0,1)$, and a constant $\theta\in[0,1)$ such that for alll $(x,y)\in R^2$, $t\in(0,1)$ either*

$$|f(t,t^{\alpha-2}x,y)|\le a(t)|x|+b(t)|y|+c(t)|y|^\theta+r(t) \tag{2.4}$$

or else

$$|f(t,t^{\alpha-2}x,y)|\le a(t)|x|+b(t)|y|+c(t)|x|^\theta+r(t). \tag{2.5}$$

(B) *there exists a constant $M>0$ such that $|x|>M$ implies*

$$\int_0^1(f(s,s^{\alpha-2}x,y)+e(s))ds\neq\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(f(s,s^{\alpha-2}x,y)+e(s))ds.$$

(C) *there exists a constant $M^* > 0$, then either*

$$c\left[\int_0^1 [f(t, ct^{\alpha-2}, 0) + e(t)]dt - \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} [f(t, ct^{\alpha-2}, 0) + e(t)]dt\right] > 0 \tag{2.6}$$

for all $|c| > M^*$ or else

$$c\left[\int_0^1 [f(t, ct^{\alpha-2}, 0) + e(t)]dt - \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} [f(t, ct^{\alpha-2}, 0) + e(t)]dt\right] < 0 \tag{2.7}$$

for all $|c| > M^*$.

(D) $\frac{2}{\Gamma(\alpha)} \int_0^1 s^{\alpha-2} a(s)ds + \int_0^1 b(s)ds < 1$.

Then for every $e \in L^1[0,1]$ BVP(1) has at least one solution.

Proof. From Lemma 2.2, L be a Fredholm operator of index zero and let N be $L-$compact on Ω.

To apply Lemma 2.1, we should define an open bounded subset Ω of X centered at zero such that (i), (ii) and (iii) in Lemma 2.1 hold. To obtain Ω, we do three steps. The proof of this theorem is divided into four steps.

Step 1. Let $\Omega_1 = \{u \in D(L) \backslash \mathrm{Ker}L,\ Lu = \lambda Nu$ for some $\lambda \in (0,1)\}$. We prove that Ω_1 is bounded.

For $u \in \Omega_1$, we get $Lu = \lambda Nu$ and

$$\begin{cases} D_{0^+}^{\alpha} u(t) = \lambda f\left(t, u(t), D_{0^+}^{\alpha-1} u(t)\right) + e(t), \quad t \in (0,1), 1 < \alpha \leq 2, \\ \lim_{t\to 0} D_{0^+}^{\alpha-1} u(t) = 0, \\ \lim_{t\to 1} D_{0^+}^{\alpha-1} u(t) = \sum_{i=1}^{m-2} \beta_i D_{0^+}^{\alpha-1} u(\xi_i). \end{cases}$$

Now, $Nu \in \mathrm{Im}L$ implies that

$$\int_0^1 (f(s, u(s), D_{0^+}^{\alpha-1} u(s)) + e(s))ds \neq$$

$$\sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} (f(s, u(s), D_{0^+}^{\alpha-1} u(s)) + e(s))ds.$$

It follows that

$$\begin{aligned}&\int_0^1 \left[f\left(s, \frac{s^{2-\alpha}u(s)}{s^{2-\alpha}}, D_{0^+}^{\alpha-1}u(s)\right)+e(s)\right]ds\\ &\neq \\ &\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}\left[f\left(s, \frac{s^{2-\alpha}u(s)}{s^{2-\alpha}}, D_{0^+}^{\alpha-1}u(s)\right)+e(s)\right]ds.\end{aligned} \tag{$\diamond$}$$

Since $t^{2-\alpha}u(t)$ is continuous on $(0,1)$, then either there exists $t_0 \in (0,1)$ such that $t_0^{2-\alpha}|u(t_0)| \le M$ or

$$t^{2-\alpha}u(t) > M \text{ for all } t \in (0,1] \tag{$*$}$$

or

$$t^{2-\alpha}u(t) < M \text{ for all } t \in (0,1). \tag{$\star$}$$

If $(*)$ and $(\star)$ hold, together with $(\diamond)$ and (B), there exists $t_0 \in [0,1]$ such that $t_0^{2-\alpha}|u(t_0)| \le -M$. Hence we have $t_0 \in (0,1)$ such that

$$t_0^{2-\alpha}|u(t_0)| \le M. \tag{$\bullet$}$$

One sees that

$$\begin{aligned} u(t) &= I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t) + c_1t^{\alpha-1} + c_2t^{\alpha-2}\\ &= \lambda I_{0^+}^{\alpha}\left[f\left(t, u(t), D_{0^+}^{\alpha-1}u(t)\right)+e(t)\right] + c_1t^{\alpha-1} + c_2t^{\alpha-2}.\end{aligned}$$

Since

$$\lim_{t\to 0} D_{0^+}^{\alpha-1}u(t) = 0,$$

then $c_1 = 0$. On the other hand, we have

$$t^{2-\alpha}u(t) = I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t) + c_2,\ t_0^{2-\alpha}u(t_0) = I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t)|_{t=t_0} + c_2.$$

Then

$$|t^{2-\alpha}u(t)| = \left|t_0^{2-\alpha}|u(t_0)| + t^{2-\alpha}I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t) - t^{2-\alpha}I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t)\big|_{t=t_0}\right|$$

$$\le M + \left|t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}D_{0^+}^{\alpha}u(s)ds - t_0^{2-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}D_{0^+}^{\alpha}u(s)ds\right|$$

$$= M + \left|t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}dD_{0^+}^{\alpha-1}u(s) - t_0^{2-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}dD_{0^+}^{\alpha-1}u(s)\right|$$

$$= M + \left|t^{2-\alpha}\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}D_{0^+}^{\alpha-1}u(s)\Big|_0^t - t_0^{2-\alpha}\frac{(t_0-s)^{\alpha-1}}{\Gamma(\alpha)}D_{0^+}^{\alpha-1}u(s)\Big|_0^{t_0}\right.$$

$$\left.t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}D_{0^+}^{\alpha-1}u(s)ds - t_0^{2-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha-2}}{\Gamma(\alpha)}D_{0^+}^{\alpha-1}u(s)ds\right|$$

$$\le M + t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}|D_{0^+}^{\alpha-1}u(s)|ds + t_0^{2-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha-2}}{\Gamma(\alpha)}|D_{0^+}^{\alpha-1}u(s)|ds$$

$$\le M + \left(t^{2-\alpha}\int_0^t \frac{(t-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds + t_0^{2-\alpha}\int_0^{t_0}\frac{(t_0-s)^{\alpha-2}}{\Gamma(\alpha-1)}ds\right)\sup_{t\in(0,1)}|D_{0^+}^{\alpha-1}u(t)|$$

$$\le M + \frac{2}{\Gamma(\alpha)}\sup_{t\in(0,1)}|D_{0^+}^{\alpha-1}u(t)|.$$

If (2.4) holds, then

$$|D_{0^+}^{\alpha-1}u(t)| = \left|D_{0^+}^{\alpha-1}u(t)|_{t=0} + \int_0^t D_{0^+}^{\alpha}u(s)ds\right|$$

$$= \left|\lambda\int_0^t \left[\, f(s,u(s),D_{0^+}^{\alpha-1}u(s)) + e(s)\right] ds\right|$$

$$\le \int_0^1 \left|\, f(s,u(s),D_{0^+}^{\alpha-1}u(s)) + e(s)\right| ds$$

$$\le \int_0^1 |f(s,s^{\alpha-2}s^{2-\alpha}u(s),D_{0^+}^{\alpha-1}u(s))|ds + \int_0^1 |e(s)|ds$$

$$\le \int_0^1 s^{\alpha-2}a(s)|s^{2-\alpha}u(s)|ds + \int_0^1 b(s)|D_{0^+}^{\alpha-1}u(s))|ds$$

$$+ \int_0^1 c(s)|D_{0^+}^{\alpha-1}u(s))|^{\theta}ds + \int_0^1 |r(s)|ds + \int_0^1 |e(s)|ds$$

$$\le \int_0^1 s^{\alpha-2}a(s)ds\sup_{t\in(0,1)}|t^{2-\alpha}u(t)| + \int_0^1 b(s)ds\sup_{t\in(0,1)}|D_{0^+}^{\alpha-1}u(t))|$$

$$+ \int_0^1 c(s)ds\left(\sup_{t\in(0,1)}|D_{0^+}^{\alpha-1}u(t))|\right)^{\theta} + \int_0^1 |r(s)|ds + \int_0^1 |e(s)|ds$$

$$\le \int_0^1 s^{\alpha-2}a(s)ds\left(M + \frac{2}{\Gamma(\alpha)}\sup_{t\in(0,1)}|D_{0^+}^{\alpha-1}\right)$$

$$+ \int_0^1 b(s)ds\sup_{t\in(0,1)}|D_{0^+}^{\alpha-1}u(t))|$$

$$+ \int_0^1 c(s)ds\left(\sup_{t\in(0,1)}|D_{0^+}^{\alpha-1}u(t))|\right)^{\theta} + \int_0^1 |r(s)|ds + \int_0^1 |e(s)|ds$$

$$= \left(\tfrac{2}{\Gamma(\alpha)} \int_0^1 s^{\alpha-2} a(s)ds + \int_0^1 b(s)ds\right) \sup_{t\in(0,1)} |D_{0^+}^{\alpha-1} u(t))|$$

$$+ \int_0^1 c(s)ds \left(\sup_{t\in(0,1)} |D_{0^+}^{\alpha-1} u(t))|\right)^\theta$$

$$+M \int_0^1 s^{\alpha-2} a(s)ds + \int_0^1 |r(s)|ds + \int_0^1 |e(s)|ds.$$

Then

$$\begin{aligned}
&\sup_{t\in(0,1)} |D_{0^+}^{\alpha-1} u(t))| \\
&\le \left(\frac{2}{\Gamma(\alpha)} \int_0^1 s^{\alpha-2} a(s)ds + \int_0^1 b(s)ds\right) \sup_{t\in(0,1)} |D_{0^+}^{\alpha-1} u(t))| \\
&+ \int_0^1 c(s)ds \left(\sup_{t\in(0,1)} |D_{0^+}^{\alpha-1} u(t))|\right)^\theta \\
&+M \int_0^1 s^{\alpha-2} a(s)ds + \int_0^1 |r(s)|ds + \int_0^1 |e(s)|ds.
\end{aligned}$$

It follows from (D) and $\theta \in [0, 1)$ that there exists a constant $M_1 > 0$ such that

$$\sup_{t\in(0,1)} |D_{0^+}^{\alpha-1} u(t))| \le M_1. \tag{2.8}$$

Then

$$\sup_{t\in(0,1)} t^{2-\alpha} |u(t)| \le M + \frac{2}{\Gamma(\alpha)} M_1.$$

Hence

$$||u||| \le \max\left\{M_1,\ M + \frac{2}{\Gamma(\alpha)} M_1\right\} =: M_2. \tag{2.9}$$

It follows that Ω_1 is bounded.

If (2.5) holds, similarly to above discussion, we can that there exists $M_2 > 0$ such that (2.9) holds. It follows that Ω_1 is bounded too.

Step 2. Let $\Omega_2 = \{x \in \mathrm{Ker}L :\ Nx \in \mathrm{Im}L\}$. We prove that Ω_2 is bounded.

For $x \in \Omega_2$, then $x(t) = ct^{\alpha-2}$, and

$$Nx(t) = f(t, ct^{\alpha-2}, 0) + e(t).$$

So

$$\int_0^1 [f(t, ct^{\alpha-2}, 0) + e(t)]dt = \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} [f(t, ct^{\alpha-2}, 0) + e(t)]dt. \tag{2.10}$$

From (B), we get that $|ct^{\alpha-2}| \le M$. Then $|c| \le M$. This shows Ω_2 is bounded.

Step 3. We prove that either

$$\Omega_3 = \{x \in \text{Ker } L : \ \lambda \wedge x + (1-\lambda)QNx = 0, \ \lambda \in [0,1]\}$$

or

$$\Omega_3 = \{x \in \text{Ker } L : \ -\lambda \wedge x + (1-\lambda)QNx = 0, \ \lambda \in [0,1]\}$$

is bounded.

If (2.6) holds for all $|c| > M^*$, let

$$\Omega_3 = \{x \in \text{Ker } L : \ \lambda \wedge x + (1-\lambda)QNx = 0, \ \lambda \in [0,1]\},$$

where $\wedge$ is the isomorphism given by $\wedge(ct^{\alpha+k-1}) = ct^{\alpha-2}$. We prove that Ω_3 is bounded.

For $x(t) = ct^{\alpha-2} \in \text{Ker } L$, one sees that

$$\begin{aligned} -\lambda ct^{\alpha+k-1} &= \frac{(1-\lambda)(\alpha+k)t^{\alpha+k-1}}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}} \left(\int_0^1 [f(t, ct^{\alpha-2}, 0) + e(t)]dt \right. \\ &\quad \left. - \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} [f(t, ct^{\alpha-2}, 0) + e(t)]dt \right). \end{aligned}$$

Then

$$\begin{aligned} -\lambda c &= \frac{(1-\lambda)(\alpha+k)}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}} \left(\int_0^1 [f(t, ct^{\alpha-2}, 0) + e(t)]dt \right. \\ &\quad \left. - \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} [f(t, ct^{\alpha-2}, 0) + e(t)]dt \right). \end{aligned}$$

If $\lambda = 1$, then $c = 0$. If $\lambda \in [0,1)$, and $|c| > M^*$, we get

$$\begin{aligned} 0 \ge -\lambda c^2 &= \frac{(1-\lambda)(\alpha+k)}{1-\sum_{i=1}^{m-2}\beta_i\xi_i^{\alpha+k}} \left(a \int_0^1 [f(t, ct^{\alpha-2}, 0) + e(t)]dt \right. \\ &\quad \left. -c \sum_{i=1}^{m-2} \beta_i \int_0^{\xi_i} [f(t, ct^{\alpha-2}, 0) + e(t)]dt \right) \\ &> 0, \end{aligned}$$

a contradiction. Hence $|c| \le M^*$. Then Ω_3 is bounded.

If (2.7) holds for all $|c| > M^*$, let

$$\Omega_3 = \{x \in \text{Ker } L : \ \lambda \wedge x - (1-\lambda)QNx = 0, \ \lambda \in [0,1]\},$$

where $\wedge$ is the isomorphism given by $\wedge(ct^{\alpha+k-1}) = ct^{\alpha-2}$. We prove that Ω_3 is bounded.

Step 4. We shall show that all conditions of Lemma 2.3 are satisfied.

Set Ω be a open bounded subset of X centered at zero such that $\Omega \supset \cup_{i=1}^{3}\overline{\Omega_i}$. By Lemma 2.2, L is a Fredholm operator of index zero and N is $L-$compact on $\overline{\Omega}$. By the definition of Ω, we have

(i) $Lx \neq \lambda Nx$ for $x \in (D(L) \setminus \text{Ker}L) \cap \partial\Omega$ and $\lambda \in (0,1)$;
(ii) $Nx \notin \text{Im}L$ for $x \in \text{Ker}L \cap \partial\Omega$.
(iii) $\deg(QN|_{\text{Ker}L}, \ \Omega \cap \text{Ker}L, 0) \neq 0$. In fact, let $H(x,\lambda) = \pm\lambda \wedge x + (1-\lambda)QNx$. According the definition of Ω, we know $H(x,\lambda) \neq 0$ for $x \in \partial\Omega \cap \text{Ker}L$, thus by homotopy property of degree,

$$\begin{aligned}\deg(QN|_{\text{Ker}L}, \Omega \cap \text{Ker}L, 0) &= \deg(H(\cdot,0), \Omega \cap \text{Ker}L, 0)\\ &= \deg(H(\cdot,1), \Omega \cap \text{Ker}L, 0)\\ &= \deg(\wedge, \Omega \cap \text{Ker}L, 0) \neq 0.\end{aligned}$$

Thus by Lemma 2.1, $Lx = Nx$ has at least one solution in $D(L) \cap \overline{\Omega}$, which is a solution of BVP(1.1). The proof is complete. □

Remark 2.4. Theorem 2.3 generalizes and improves the main result (Theorem 3.8) in [10]. In fact, when $\alpha = 2$, BVP(1.1) becomes

$$\begin{cases} u''(t) = f(t, u(t), u'(t)) + e(t), t \in (0,1),\\ u'(0) = 0,\\ u'(1) = \sum_{i=1}^{m-2} \beta_i u'(\xi_i).\end{cases}$$

(i) Firstly the assumptions $\sum_{i=1}^{m} \beta_i = 1$ and $\sum_{i=1}^{m} \beta_i\xi_i \neq 1$ in [10: Theorem 3.8] are cancelled in Theorem 2.3.
(ii) Secondly, the assumption (D) is

$$2\int_0^1 a(s)ds + \int_0^1 b(s)ds < 1 \text{ when } \alpha = 2.$$

It is weaker than

$$\int_0^1 a(s)ds + \int_0^1 b(s) < \frac{1}{2} \text{ supposed in } [10: \text{ Theorem 3.8}].$$

(iii) the other assumptions in Theorem 2.3 are exactly same to those ones of Theorem 3.8 in [10].

REMARK 2.5. Similarly to Theorem 2.3, for the following BVP

(2.11)
$$\begin{cases} D_{0+}^{\alpha}u(t) = f\left(t, u(t), D_{0+}^{\alpha-1}u(t)\right) + e(t), \quad t \in (0,1), 1 < \alpha \leq 2, \\ \lim_{t\to 0} D_{0+}^{\alpha-1}u(t) = \sum_{i=1}^{m-2} \beta_i D_{0+}^{\alpha-1}u(\xi_i), \\ \lim_{t\to 1} D_{0+}^{\alpha-1}u(t) = 0, \end{cases}$$

we can establish existence result for solutions. The details are omitted.

3. An example

Now, we present an example, which can not be covered by known results, to illustrate Theorem 2.3.

EXAMPLE 3.1. Consider the boundary value problem

$$\begin{cases} D_{0+}^{\frac{3}{2}}(t) = \frac{1}{24}x(t) + \frac{1}{24}\sin\left(D_{0+}^{\frac{1}{2}}x(t)\right) \\ +3\sin\left(D_{0+}^{\frac{1}{2}}x(t)\right)^{\frac{1}{3}} + 1 + \cos^2 t, \\ \lim_{t\to 0} D_{0+}^{\frac{1}{2}}u(t) = 0, \\ \lim_{t\to 1} D_{0+}^{\frac{1}{2}}u(t) = \frac{1}{2}D_{0+}^{\frac{1}{2}}x\left(\frac{1}{4}\right) + \frac{1}{2}D_{0+}^{\frac{1}{2}}x\left(\frac{1}{2}\right). \end{cases} \tag{3.1}$$

Corresponding to BVP(1.1), $\alpha = \frac{3}{2}$ and

$$\xi_1 = \frac{1}{4},\ \xi_2 = \frac{1}{2},\ \beta_1 = \frac{1}{2},\ \beta_2 = \frac{1}{2},$$

$$f(t,x,y) = \frac{1}{24}x + \frac{1}{24}\sin y + 3\sin y^{\frac{1}{3}}, \quad e(t) = 1 + \cos^2 t.$$

(A) choose $a(t) = \frac{1}{24}$, $b(t) = \frac{1}{24}$, $c(t) = 3$, $r(t) = 0$, $\theta = \frac{1}{3}$, then for all $(x,y) \in R^2$, $t \in (0,1)$ either

$$|f(t,x,y)| \leq a(t)|x| + b(t)|y| + c(t)|y|^{\theta} + r(t). \tag{3.2}$$

(B) choose $M = 122$, it is easy to find that

$$f(t,x,y) + e(t) \geq \frac{1}{24}x - \frac{1}{24} - 3 + 1 = \frac{x-49}{24} > \frac{1}{24} \text{ if } x > M,$$

then

$\int_0^1 (f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds - \frac{1}{2}\int_0^{\frac{1}{4}}(f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds$

$-\frac{1}{2}\int_0^{\frac{1}{2}}(f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds$

$=\frac{1}{2}\int_{\frac{1}{4}}^1 (f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds$

$+\frac{1}{2}\int_{\frac{1}{2}}^1 (f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds$

$>0.$

It is easy to find that

$$f(t,x,y)+e(t)\le \frac{1}{24}x+\frac{1}{24}+3+2=\frac{x+120}{24}<\frac{-2}{24} \text{ if } x<-M,$$

then

$\int_0^1 (f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds - \frac{1}{2}\int_0^{\frac{1}{4}}(f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds$

$-\frac{1}{2}\int_0^{\frac{1}{2}}(f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds$

$=\frac{1}{2}\int_{\frac{1}{4}}^1 (f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds$

$+\frac{1}{2}\int_{\frac{1}{2}}^1 (f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds$

$<0.$

Then $u\in D(L)$, if $|u(t)|>M$ for all $t\in[0,1]$, it holds that

$$\int_0^1 (f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds \neq$$

$$\sum_{i=1}^{m-2}\beta_i\int_0^{\xi_i}(f(s,u(s),D_{0^+}^{\alpha-1}u(s))+e(s))ds.$$

(C) since

$$f(t,ct^{\alpha-2},0)+e(t)=\frac{1}{24}ct^{\alpha-2}+1+\cos^2 t,$$

we get

$$A = c\left[\int_0^1 [f(t, ct^{\alpha-2}, 0) + e(t)]dt - \frac{1}{2}\int_0^{\frac{1}{4}} [f(t, ct^{\alpha-2}, 0) + e(t)]dt\right.$$

$$\left. - \frac{1}{2}\int_0^{\frac{1}{2}} [f(t, ct^{\alpha-2}, 0) + e(t)]dt\right]$$

$$= c\left[\frac{1}{2}\int_{\frac{1}{4}}^1 [f(t, ct^{\alpha-2}, 0) + e(t)]dt + \frac{1}{2}\int_{\frac{1}{2}}^1 [f(t, ct^{\alpha-2}, 0) + e(t)]dt\right]$$

$$= c\left[\frac{1}{2}\int_{\frac{1}{4}}^1 \left(\frac{1}{24}ct^{\alpha-2} + 1 + \cos^2 t\right) dt + \frac{1}{2}\int_{\frac{1}{2}}^1 \left(\frac{1}{24}ct^{\alpha-2} + 1 + \cos^2 t\right)]dt\right]$$

$$= \frac{1}{48}\left(2 - \frac{1}{4^{\alpha-1}} - \frac{1}{2^{\alpha-1}}\right)c^2 + \frac{1}{2}\int_{\frac{1}{4}}^1 \left(1 + \cos^2 t\right) dt + \frac{1}{2}\int_{\frac{1}{2}}^1 \left(1 + \cos^2 t\right) dtc.$$

It is easy to see that there exists $M^* > 0$ such that $A > 0$ for all $|c| > M^*$

(D) $\frac{2}{\Gamma(\alpha)}\int_0^1 \frac{a(s)}{s^{2-\alpha}}ds + ||b||_1 < 1.$

It follows from Theorem 2.3 that BVP(3.1) has at least one solution x.

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Department of Mathematics
Hunan Institute of Science and Technology
Yueyang 414000, P R China
E-mail: liuyuji888@sohu.com

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Department of Mathematics
Shaoyang University
Shaoyang Hunan 422000, P R China
E-mail: liuxingyuan@sohu.com